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CORRECTION FUNCTION FOR THE SERIES $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)}$

DR KUMARI SREEJA S NAIR
Associate Professor of Mathematics
Govt. Arts College, Kariavattom, Thiruvananthapuram, India

Abstract

In this paper we give a rational applying a correction function to the series. function certainly improves the value of sum of the series and gives a approximation to it.

1. Introduction

Commenting on the Lilavati rule for finding the value of circumference of a circle from its diameter, the commentator series for computing the circumference from the diameter. One such series attributed to illustrious mathematician Madhava of 14-th century is

$$C = \frac{4d}{1} - \frac{4d}{3} + \frac{4d}{5} - \cdots + \frac{4d}{2n-1} \mp \frac{4d\left(\frac{2n}{2}\right)}{(2n)^2+1},$$

where + or − indicates that n is odd or even and C is the circumference of a circle of diameter d.

Key Words : *Correction function, Error function.*

2. Approximation of the Series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)}$

The alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)}$ is convergent and converges to 2log2 -1.

$$2log2 - 1 = \frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - c \cdots + \cdots$$

If R_n denotes the remainder term afrer n terms of the series, then $R_n - (-1)^n G_n$ where G_n is the correction function after n terms of the series.

Theorem 1 : The correction function for the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)}$ is $G_n = \frac{1}{2(n+1)2^2+1^2}$.

Proof : If G_n denotes the correction function for the series after n terms, then it follows that $G_n + G_{n+1} - \frac{1}{(n+1)(n+2)}$.

The error function is $E_n - G_n + G_{n+1} - \frac{1}{(n+1)(n+2)}$.

We may choose G_n in such a way that $|E_n|$ is a minimumm function of n.

Let $G_n = \frac{1|}{(2n^2+6n+4)-(r_1 n+r_2)}$.

For a fixed n and for any $r_1, r_2 \subset R$, choose

$$G_n(r_1, r_2) - \frac{1}{(2n^2 + 6n + 4) - (r_1 n + r_2)}.$$

Then the error function is

$$E_n(r_1, r_2) = G_n(r_1, r_2) + G_{n+1}(r_1, r_2) - \frac{1}{(n+1)(n+2)}$$

is a rational function of r_1 and r_2. i.e.

$$E_n(r_1, r_2 - \frac{N_n(r_1, r_2)}{D_n(r_1, r_2)}.$$

$D_n(r_1, r_2) \approx 4n^6$, which kis a maximum for large values of n.

$|N_n(r_1, r_2)|$ is a minimum function of n for $r_1 - 2$ and $r_2 = 1$ and the minimum value is 3.

Thus $|E_n(r_1.r_2)|$ is a minimum function of n for $r_1 = 2$ and $r_2 = 1$.

Thus for $r_1 - 2$ and $r_2 = 1$ both G_n and E_n are functions of a single variable n.

Hence the correction function for the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)}$ is

$$G_n - \frac{1}{(2n^2 + 6n + 4) - (2n + 1)} = \frac{1}{2(n+1)^2 + 1^2}.$$

The corresponding error function is

$$|E_n| - \frac{1^2(1-3)}{\{2(n+1)^2+1^2\}\{2(n+2)^2+1^2\}\}\{(n+1)(n+2)\}}.$$

Hence the proof.

3. Remark

Clearly $G_n < \frac{1}{(n+1)(n+2)}$, absolute value of $(n+1)^{th}$ term.

Theorem 2 : The correction functions for series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)}$ follow an infinite continued fraction

$$\frac{1}{(1(n+1)^2+1^2)} - \frac{1^2(1.3)}{(2(n+1)^2+3^2)} + \frac{2^2(3.5)}{(2(n+1)^2+5^2)} + \frac{3^2(5.7)}{(2(n+1)^2+7^2)+\cdots}.$$

Proof : In Theorem 1, we have showed that the correction function for this series is $G_n = \frac{1}{2(n|1)^2|1^2}$ and the corresponding error function is

$$|E_n| = \frac{1^2(1.3)}{\{2(n+1)^2+1^2\}\{2(n+2)^2+1^2\}\{(n+1)(n+2)\}}.$$

We may rename this correction function as the first order correction function and denote it as $G_n[1] = \frac{1}{2(n+1)^2+1^2}$ and the error function is

$$|E_n[1]| = \frac{1^2(1.3)}{\{2(n+1)^2+1^2\}\{2(n+2)^2+1^2\}\{(n+1)(n_2)\}}.$$

For further reducing error functioin we may add fractions of correction divisor to the correction divisor itself.

Choose $G_n[2] = \frac{1}{\{2(n+1)^2 1^2\} + \frac{A_1}{\{2(n+1)^2+1^2\}+x\}}}$ where A_1 and X are any two real numbers.

Then it can be verified that absolute value of the error function is a minimum function of n for $A_1 = -3$ and $x = 8$.

Thus $G_n[2] = \frac{1}{\{2(n+1)^2+1^2\}\frac{1^2(1/3)}{\{2(n+1)^2+3^2\}}}$ and it is the second order correction function.

Now for reducing error choose

$$G_n[3] = \frac{1}{\{2n+1_{2}{+1^2}\} - \frac{1^2(1.3)}{\{2(n+1)^2+2^2\}+\frac{A_2}{\{2(n+1)^2+2^2\}+x}}}.$$

It can be proved that $|E_n|$ is minimum for $A_2 = -60$ and $x = 16$.

Thus the third order correctiion function is

$$G_n[3] = \frac{1}{\{2(n+1)^2+1^2\}\frac{1^2(1.3)}{\{2(n+1)^2+3^2\}\frac{2^2(3.5)}{\{2(n+1)^2+5^2\}}}}.$$

Similarly the fourth order correction function is

$$G_n[4] = \frac{1}{\{2(n+1)^2+1^2\}}\frac{1^2(1.3)}{\{2(n+1)^2+3^2\} - \frac{2^2(3.5)}{\{2(n+1)^2+5^2\}\frac{3^2(5.2)}{\{2(n+1)^2+7^2\}}}}.$$

In general, the $?k^{th}$ order correctiion function is

$$\begin{aligned} G_n(k) \quad = \quad & \frac{1}{\{2(n+1)^2+1\}} - \frac{1^2(1.2)}{\{2(n+1)^2+3^2\}} - \frac{2^2(2.5)}{\{2(n+1)^2+5^2\}} \\ & - \frac{2^2(5.7)}{\{2(n+1)^2+7^2\}} - \cdots - \frac{(k-1)^2(2k-2)(2k-1)}{\{2(n+1)^2+(2k-1)^2\}} \end{aligned}$$

Continuing this process we get the correction functions follow an infinite continued fractioin pattern

$$\frac{1}{(2(n+1)^2+1^2)} - \frac{1^2(1.3)}{(2(n+1)^2+3^2)} - \frac{2^2(3.5)}{(2(n+1)^2+5^2)} - \frac{3^2(5.7)}{(2(n+1)^2+7^2)\cdots}.$$

4. Application

For $n = 10$, the series approximation after applying the correction functions are given below.

We 2log2 - 1 = 0.3862943611, using a calculator..

Correction function	$S_n + (-1)^n G)_n$
Without correction function	0.3821789321
$G_n[1]$	0.3863283098
$G_n[2]$	0.3862943611
$G_n[3]$	0.3862943611

5. Conclusion

The correction functions are the successive continued fraction. Thus the accuracy can be improved.

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